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ON THE EXISTENCE AND CLASSIFICATION OF EXTENSIONS OF ACTIONS ON SUBMANIFOLDS OF DISKS AND SPHERES

BY

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Dedicated to Deane Montgomery

ABSTRACT. Given a G-action $\psi: G \times W \to W$ and an embedding $W \subset D^n$, when is it possible to find a G-action $\phi: G \times D^n \to D$ such that $D^n - W$ is G-free?

Sufficient conditions of cohomological nature for the existence of such extensions are given and the extensions are classified. This leads to the characterization of the stationary point sets and classification of semifree actions on disks up to G-diffeomorphism under suitable dimension hypotheses.

Introduction. In his landmark paper, Jones [**J**] (in 1970) showed the following very strong converse to the fixed point theorem of P. A. Smith. A manifold V^n which is a $\mathbf{Z}/p\mathbf{Z}$-homology disk is the fixed set for an action of $\mathbf{Z}/p\mathbf{Z}$ on a contractible polyhedron, and it is the fixed set of a smooth $\mathbf{Z}/p\mathbf{Z}$-action on D^{n+k}, $k > n + 2$, provided we have such an action on a regular neighborhood of $V \subset D^{n+k}$.

The combinatorial result was greatly generalized by Oliver [**O**], whose work was refined still further by Assadi [**As1**] and Oliver-Petrie [**OP**]. These papers produce combinatorial or algebraic K-theoretic conditions that a given action of a finite group G on a complex L extends to a G-action on a contractible complex $Q \supset L$.

In this paper, we consider the smooth problem: given a smooth action of G on a manifold W^n, under what circumstances can we extend the action to an action either

(a) on a euclidean space $\mathbf{R}^n$ in which W^n is properly embedded, or

(b) on a disk $D^n \supset W^n$, such that G acts freely on $\mathbf{R}^n - W^n$ or $D^n - W^n$?

The problems (a) and (b) depend first on being able to extend the action to (a) a finite-dimensional contractible complex, or (b) a finite contractible complex, for which we have the analysis of Assadi [**As1**], and in addition there is a tangential obstruction, a function of the tangent bundle of W^n with its G-action. With various "general position" hypotheses weaker than that in [**J**], these obstructions are the only ones.

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The tangential obstruction can be thought of as the obstruction to finding a bundle ξ over the classifying space B_G of G whose pull-back over the free orbits of ∂W agrees with the tangent bundle over a large skeleton or, more elegantly, in a formulation of Dovermann [**Dov**], that the 'extended' tangent bundle $\hat{\tau}_W$ over $W \times_G E_G$ (E_G the universal free contractible G-space) comes from ξ.

In §2, we give a sufficient condition for the vanishing of this tangential obstruction in terms of the vanishing of some cohomology groups of G with coefficients in $\widetilde{KO}(W)$. We also give a sufficient condition to ensure Smith acyclicity in terms of group cohomology, but we show that these cohomological conditions are not necessary. We then apply the results to the case of semifree G-actions. We obtain the appropriate generalization of Jones [**J**] to such a general finite G, as well as extending the Jones result to actions which do not necessarily preserve orientation and to some special cases where the tangential part of the proof of [**J**] is inadequate (§4). We also show that such actions extending the given action on a regular neighborhood of the fixed set are classified by Whitehead torsion (§3).

In §0, we recall some combinatorial results from [**As1**] and give the technical results in §1, improving some of the dimension restrictions in §2.

We assume familiarity with the usual mechanism of smooth G-actions, as in [**Bre2**]. We also refer routinely to G-regular neighborhoods, as in [**I**].

Finally, we note that there is considerable overlap of our results with those of Dovermann [**Dov**].

REMARK. Many of our arguments do not depend on the smoothness of the action and, in fact, the main existence and classification theorems hold for any category of G-manifolds in which a reasonable theory of equivariant regular neighborhoods exists. Such equivariant neighborhoods are not required to be unique for the existence of extensions.

0. Combinatorial preliminaries. In this section we summarize some background material. The reader is referred to [**As1**] for further details and examples. In the sequel, all G-spaces are G-CW-complexes and are referred to as G-complexes for short. The results below hold for smooth and PL actions, as well as the special case of topological G-actions on manifolds in which a reasonable theory of equivariant regular neighborhoods exists. By a PL G-action on a PL-manifold M it is meant that elements of G act by simplicial maps for some appropriate simplicial subdivision of M. The second baricentric subdivision of M gives M the structure of a G-CW-complex (cf. [**Bre1**]). By a result of Illman, a smooth G-manifold has the structure of a G-CW-complex, for G finite [**I**].

For a G-manifold (smooth and PL), *the singular set* is defined by $\mathscr{S}(X) = \bigcup_{H \subseteq G, H \neq 1} X^H$, and $X - \mathscr{S}(X)$ is called the *free stratum* of the action. By a straightforward induction on the number of orbit types, it is seen that $\mathscr{S}(X)$ has regular neighborhood (smooth or PL) which is invariant under the action.

0.1. DEFINITION. Let $X \subset Y$ and let ϕ: $G \times X \to X$ and ψ: $G \times Y \to Y$ be actions such that $\psi | G \times X = \phi$. We call ψ a *free extension* of ϕ if $Y - X$ is a free G-space. For short, Y may be called a free extension of X.

0.2. DEFINITION. A G-complex X is called *Smith acyclic* if, for each prime power order subgroup $H \subseteq G$, $|H| = p^r \neq 1$, $\overline{H}_*(X^H; \mathbf{Z}/p\mathbf{Z}) = 0$.

This definition is motivated by the well-known theorem of P. A. Smith stating that for a G-action on a finite-dimensional acyclic G-complex X, $\overline{H}_*(X^H; \mathbf{Z}/p\mathbf{Z}) = 0$ for every p-subgroup H. So the singular set of an action on a finite-dimensional contractible complex is Smith acyclic. The natural question is whether the singular set of a Smith acyclic complex can be realized as the singular set of a contractible G-complex. This is true for finite-dimensional G-complexes, and in the category of finite G-complexes an affirmative answer depends on the vanishing of an obstruction in $\tilde{K}_0(\mathbf{Z}G)$, which we will discuss below.

Suppose we would like to find a free extension ψ: $G \times X \to X$ of a given action ϕ: $G \times X_0 \to X_0$, where (X_0, ϕ) is a finite-dimensional Smith acyclic G-complex. We proceed by adding equivariant G-cells of dimension $\leqslant 2$ to X_0 to obtain a 1-connected G-CW-complex X_1 containing X_0 and extending the action on X_0 freely. Inductively, we may assume that we have a k-connected G-CW-complex X_k which extends X_0 freely and such that $\overline{H}_{k+1}(X_k)$ is the only nonvanishing homology.

0.3. PROPOSITION. *If X_0 is Smith acyclic, $X_k \supset X_0$ such that G acts freely on $X_k - X_0$, and $H_{k+1}(X_k)$ is the only nonvanishing reduced homology group, then $H_{k+1}(X_k)$ is cohomologically trivial. If, in addition, $H_{k+1}(X_k)$ is free over $\mathbf{Z}$, then it is $\mathbf{Z}G$-projective, and if X_k is finite, then the class $(-1)^{k+1}[H_{k+1}(X_k)] \in \tilde{K}_0(\mathbf{Z}G)$ depends on $\mathscr{S}(X_0)$.*

See [**As1**, Propositions I.1.6 and I.2.1], as well as [**O**], for a proof.

In general, by the Eilenberg trick, if $H_{k+1}(X_k)$ is $\mathbf{Z}G$-projective, then there is an infinitely generated free module F such that $H_{k+1}(X_k) + F$ is free over $\mathbf{Z}G$. So we can add an infinite number of G-free $(k+1)$-cells with trivial attaching maps to get X_k' with $H_{k+1}(X_k')$ free.

If X_k is finite, then $(-1)^{k+1}[H_{k+1}(X_k)]$ is independent of the free extension X_k. If $[H_{k+1}(X_k)] = 0$ in $\tilde{K}_0(\mathbf{Z}G)$, then $H_{k+1}(X_k)$ is stably free over $\mathbf{Z}G$. By adding free orbits of k-cells with null-homotopic attaching maps, a new free extension X_k' is obtained, such that $H_{k+1}(X_k')$ is actually free over $\mathbf{Z}G$.

A free basis of $H_{k+1}(X_k') \cong \pi_{k+1}(X_k')$ over $\mathbf{Z}G$ can be represented by maps α: $S^{k+1} \to X_k'$, and adding free $(k+2)$-cells using these attaching maps kills H_{k+1} without introducing any new homology. The result of this operation is a contractible G-complex which is a free extension of X_0 and which is finite when X is finite, and the appropriate projective class obstruction vanishes.

Let $\sigma(X_0) = (-1)^{n+1}[H_{n+1}(X_{k+1})] \in \tilde{K}_0(\mathbf{Z}G)$ denote this "homological obstruction" in the case where X_k is finite. Thus the above observation and Proposition 0.3 prove the following

0.4. PROPOSITION. *Let X_0 be a Smith acyclic finite-dimensional G-complex. There is a finite-dimensional contractible G-CW-complex which is a free extension of X_0. If X_0 is finite, there is a contractible finite G-CW-complex which is a free extension of X_0 if and only if $\sigma(X_0) \in \tilde{K}_0(\mathbf{Z}G)$ vanishes.*

See [**As1**, Theorem II.1.4].

In the special case where the action of G on $H_*(X_0)$ is trivial and $H_*(X_0; \mathbf{Z}/|G|\mathbf{Z}) = 0$, we have a simple formula to calculate $\wp(X_0)$. Let $\mathcal{R}$ be the category of isomorphism classes of finite abelian groups of order prime to $|G|$. Then, the Swan map $\sigma_G\colon R \to \tilde{K}_0(\mathbf{Z}G)$ is defined as follows. Let $0 \to P \to (\mathbf{Z}G) \overset{r}{\to} A \to 0$ be a short exact sequence, where A is regarded as a $\mathbf{Z}G$-module with the trivial G-action. Then, P is cohomologically trivial and $\mathbf{Z}$-free, hence it is $\mathbf{Z}G$-projective by a result of Rim [**R**, Theorem 4.12]. By a lemma of Schanuel [**S**, Lemma 1.3], $[P] \in \tilde{K}_0(\mathbf{Z}G)$ is well defined. Then we define $\sigma_G(A) = [P] \in \tilde{K}_0(\mathbf{Z}G)$. If A is any $\mathbf{Z}G$-module which is cohomologically trivial, we can still apply the same argument to define an element $\sigma'_G(A) \in \tilde{K}_0(\mathbf{Z}G)$ independent of the short exact sequence chosen above. The map σ' induces an isomorphism from the reduced Grothendieck group of cohomologically trivial $\mathbf{Z}G$-modules to $\tilde{K}_0(\mathbf{Z}G)$.

0.5. PROPOSITION. *Let X_0 be a finite G-complex such that $\overline{H}_*(X_0; \mathbf{Z}/|G|\mathbf{Z}) = 0$; then $\wp(X_0)$ is well defined and $\wp(X_0) = \sum_{i>0}(-1)^i\sigma'_G(H_i(X_0))$. If G acts trivially on $H_*(X_0)$, then*

$$\wp(X_0) = \sum_{i>0} (-1)^i \sigma_G(H_i(X_0)).$$

The proof of this follows from [**As1**, Chapter II, in particular Propositions II.8.2 and II.6.6]. A direct argument is given below. Let X_k be a free extension of X_0 obtained by adding cells of dimension less than $k+2$, so that X_k is k-connected. Induction and a Mayer-Vietoris sequence argument shows that the sequence

$$(*)\quad 0 \to H_{k+1}(X_{k-1}) \to H_{k+1}(X_k) \to C_{k+1}(X_k, X_{k+1}) \overset{\delta}{\to} H_k(X_{k-1}) \to 0$$

is exact. By induction, $H_k(X_{k-1})$ is cohomologically trivial over $\mathbf{Z}G$, and since $C_{k+1}(X_k, X_{k-1})$ is a free $\mathbf{Z}G$-module, it follows that $\operatorname{Ker}(\delta)$ is $\mathbf{Z}G$-projective. Thus, $H_i(X_k) \cong H_i(X_{k-1})$ for $i \neq k, k+1$, and $H_{k+1}(X_k) \cong H_{k+1}(X_{k-1}) \oplus \operatorname{Ker}(\delta)$ and $H_k(X_k) = 0$, so that $\overline{H}_*(X_k)$ is also cohomologically trivial. The above exact sequence $(*)$ and induction show that the quantity $\sum_{i>0}(-1)^i\sigma'_G(H_i(X_k))$ does not depend on k. For sufficiently large k, say $k \geqslant \dim X_0$, $H_{k+1}(X_k)$ is the only nonvanishing (reduced) homology group, and

$$\sum_{i>0} (-1)^i \sigma'_G(H_i(X_k)) = (-1)^{k+1}[H_{k+1}(X_k)] = \wp(X_0)$$

by Proposition 0.3 and the subsequent discussion. For the case that G acts trivially, notice that

$$\sum_{i>0} (-1)^i \sigma'_G(H_i(X_0)) = \sum_{i>0} (-1)^i \sigma_G(H_i(X_0))$$

(let $k = 0$ in the corresponding formula for X_k).

The fact that X_0 is necessarily Smith acyclic is easy to see. One further remark is helpful in the case of free extensions of actions from submanifolds of disks. Let $W^n \subset D^n$ be a codimension-zero submanifold, and suppose that W is an effective G-manifold. Suppose that we would like to construct a free extension of the action on W^n to D^n. The following necessary and sufficient condition proves useful. We state it under some dimension restrictions which may be relaxed slightly, or replaced by other conditions of similar nature.

0.6. PROPOSITION. *Let $W^n \subseteq D^n$ be as above and let $\dim W^H < n/2$ for all $1 \neq H \subset G$ and $H_i(W^n) = 0$ for $i \geqslant n/2$. Then the action on W^n extends freely to a contractible compact G-manifold $V^n \supset W^n$ if and only if*:

(i) *There exists a finite contractible X which is a free extension of W^n. This is satisfied if and only if W^n is Smith acyclic and $\sigma(W) = 0$.*

(ii) *There exists a linear G-bundle β over X such that $\beta | W^n$ is stably equivalent to $\tau(W^n)$. ($\tau(W^n)$ = the stable tangent bundle of W^n.)*

This is a special case of a more general result in [**As1**] and is obtained by the process of equivariant thickening. (Cf. [**As1**, III.2.3].)

So in the extension problem our effort will be concentrated on the construction of the G-bundle β over X.

1. Extending actions to disks. Let G be a finite group, W^n a smooth G-manifold, ϕ: $G \times W \to W$, and suppose there is a decomposition $\partial W = \partial_0 W \cup \partial_1 W$ such that G acts freely on $\partial_1 W$.

Questions. (a) Under what conditions is there a proper G-embedding of $W - \partial_0 W \subset \mathbf{R}^n$, for some smooth G action on $\mathbf{R}^n$, with G acting freely on $\mathbf{R}^n - (W - \partial_0 W)$?

(b) If W is compact, under what conditions is there a G-embedding $(W, \partial_0 W) \subset (D^n, S^{n-1})$ for some smooth G-action on the disk D^n, with G acting freely on $D^n - W^n$?

There are necessary conditions of a homological nature as reviewed in §0, namely, Smith acyclicity of W and, for (b), the vanishing of an obstruction $\sigma(W) \in \tilde{K}_0(\mathbf{Z}G)$, which ensure that W embeds in a finite contractible G-complex K, such that G acts freely on $K - W$.

If W^n embeds smoothly as a G-submanifold of some G action on $\mathbf{R}^n$, then the tangent bundle τ_W of W extends to $\tau_{\mathbf{R}^n}$; that is, τ_W extends to a G-bundle over a contractible G-complex containing W. This condition may be expressed in the following elegant statement (1.1) suggested to us by H. Dovermann (see [**Dov**]). This replaces a more awkward condition on $\tau_{\partial_0 W}$ in our earliest version.

Let E_G be a universal (i.e. contractible) G-space and, for any G-space X, let $\hat{X} = E_G \times_G X$ = the orbit of the diagonal action of G on $E_G \times X$ (the so-called Borel construction, or the balanced product). Then $\hat{X}$ is a fibre bundle over $E_G/G = B_G$ with fibre X, p: $\hat{X} \to B_G$. If ξ is a G-vector bundle over X, let $\hat{\xi}$ be the vector bundle over X obtained in this way.

1.1. PROPOSITION. *If a stable G-vector bundle ξ over X extends over some contractible G-complex $K \supset X$, then $[\hat{\xi}] = p^* z$ for some $z \in KO(B_G)$.*

PROOF. If ξ extends to ζ over K, then $\hat{\xi} = \hat{i} * \hat{\zeta}$, where i: $X \subset K$, inducing $\hat{i}$: $\hat{X} \subset \hat{K}$. Since p_1: $\hat{K} \to B_G$ is a bundle map with fibre K and since K is contractible, p_1 is a homotopy equivalence, and since $p = p_1 \circ \hat{i}$, it follows that

$$\hat{\xi} = \hat{i} * \hat{\zeta} = p^*\left(p_1^{*-1}\hat{\zeta}\right). \quad \square$$

Thus we have another tangential necessary condition for W to be a G-submanifold of D^n, i.e. $\hat{\tau}_W \in p^*KO(B_G)$.

1.2. THEOREM. *Let W^n be a compact smooth G-manifold, $\partial W = \partial_1 W \cup \partial_0 W$, where $\partial\partial_1 W = \partial\partial_0 W = \partial_1 W \cap \partial_0 W$ and G acts freely on $\partial_1 W$. Suppose that $(\partial_0 W, \partial\partial_0 W)$ is 1-connected, $(W, \partial_1 W)$ is k-connected, and $H^i(W) = 0$ for $i > k$, where $2 \leqslant k < (n-1)/2$. Then the conditions*

(1) *W is Smith acyclic,*

(2) *$\sigma(W) = 0$ in $\tilde{K}_0(\mathbf{Z}G)$, and*

(3) $[\hat{\tau}_W] \in p^*KO(B_G)$

are together necessary and sufficient for the existence of a smooth G-embedding $(W, \partial_0 W) \subset (D^n, S^{n-1})$, with G acting freely on $D^n - W$, and $D^n = W \cup$ (handles of index $\leqslant k+1$).

PROOF OF THEOREM 1.2. The necessity of the conditions (1)–(3) has been discussed above. To prove the sufficiency, assume that we are given W^n satisfying the above conditions. First, notice that we have the (homotopy) commutative diagram

$$\begin{array}{ccccc}
 & & \partial_1 W & \xrightarrow{j} & W \\
 & \swarrow q & \downarrow & & \downarrow \\
\partial_1 W/G & \leftarrow \widetilde{\partial_1 W} & = E_G \times_G \partial_1 W & \xrightarrow{\hat{j}} & E_G \times_G W = \hat{W} \\
 & \searrow \psi & \downarrow p' & & \downarrow p \\
 & & B_G & \xrightarrow{h} & B_G
\end{array}$$

in which h is a homotopy equivalence, q is the orbit map, and ψ is the classifying map. The map $\hat{j}$ is induced by the inclusion j, and p and p' are projections of the fibre bundles in the corresponding balanced products. Condition (3) applied to the above diagram implies the existence of a stable linear bundle ξ over BG such that $\psi^*\xi \cong \tau(\partial_1 W/G)$. Thus, ψ: $\partial_1 W/G \to BG$ is a prenormal map, i.e., it is covered by a bundle map b from the stable normal bundle of $\partial_1 W/G$ to a stable linear bundle ξ^{-1} over BG.

1.3. PROPOSITION. *Let W^n be a smooth connected manifold and assume that*

(1) *$\varnothing \neq \partial W = \partial_1 W \cup \partial_0 W$ with G acting freely on $\partial_1 W$, and W is connected.*

Suppose further that $(W, \partial_1 W)$ is $(k-1)$-connected, $k \leqslant (n-1)/2$ and $[\tau_{\partial_1 W/G}] = p^*(\xi)$ for some $\xi \in KO(B_G)$, with p: $\partial_1 W/G \to B_G$ the classifying map. Then there is an embedding of smooth G-manifolds $W \subset W_k$ obtained from W by adding free G-handles of index $\leqslant k$ to W along $\partial_1 W$, so $\partial_0 W \subset \partial W_k$ in case (1), *a proper embedding in case* (2) *such that W_k satisfies all the conditions of* (1.3) (*with bundle ξ, $\partial_0 W_k = \partial_0 W_k = \partial_0 W$*), *and W_k and $\partial_1 W_k$ are $(k-1)$-connected, and $H^i(W) = H^i(W_k)$ for $i > k$.*

PROOF. We may do surgery on (ψ, b), ψ: $\partial_1 W/G \to B_G$, such that if V^n is the trace of the surgery, ψ: $V^n \to B^G$, $\partial V = \partial_1 W/G \cup \partial_1 V$, then V is obtained from

$\partial_1 W/G \times [0,1]$ by adding handles of index $\leqslant k$, $\partial(\partial_1 W/G) = \partial(\partial_1 V/G)$ and $\pi_j(\psi) = \pi_j(\psi | \partial_1 v) = 0$ for $j \leqslant k \leqslant (n-1)/2$, as in [**Bro**, IV.1.13]. Note that this statement is actually proved, though only the conclusion $\pi_j(\Psi | \partial_1 V) = 0$ is stated there.

Since E_G is contractible, the universal covers $\tilde{V}$ of V and $\partial_1 \tilde{V}$ of $\partial_1 V$, are $(k-1)$-connected, and $(W, \partial_1 W)$ is $(k-1)$-connected, by hypothesis. From the Mayer-Vietoris theorem it follows that $W_k = W \cup V$ is $(k-1)$-connected (or from the Siefert-van Kampen theorem if $k = 2$). Since W_k is obtained by adding handles of index $\leqslant k$ to W, $H^i(W_k) = H^i(W)$ for $i > k$, completing the proof of 1.3. □

Returning to the proof of 1.2, in this case we get $H_i(W_k) = 0$ for $i \neq k$ and, since $H^{k+1}(W_k) = 0$, $H_k(W_k)$ is $\mathbf{Z}$-free. Hence by 0.3, $H_k(W_k)$ is $\mathbf{Z}G$-projective, since we have assumed that (by 1.2(1)) W is Smith acyclic and $W_k - W$ is G-free. By 1.2(3) and 0.3, $H_k(W_k)$ is stably $\mathbf{Z}G$-free; so, by adding more G-free k-handles along $\partial_1 W_k$ if necessary with trivial attaching maps, we may assume $H_k(W_k)$ is $\mathbf{Z}G$-free.

Since $(W_k, \partial_1 W_k)$ is k-connected, $k < (n-1)/2$ a $\mathbf{Z}G$-free basis for $\pi_k(W_k) \cong H_k(W_k)$ can be represented by a finite number of disjoint framed embedded k-spheres in $\partial_1 W_k/G$, and adding G-handles along the inverse images of these spheres in $\partial_1 W_k$ will produce a contractible manifold $U \supset W_k$, with $\partial U = \partial_0 W \cup \partial_1 U$.

Since $(\partial_0 W, \partial\partial_0 W)$ is 1-connected and $\partial_1 U$ is k-connected, ∂U is 1-connected and U is diffeomorphic to the disk D^n. By construction, $U - W$ is G-free, so the theorem follows. □

A pair (X, Y) of locally compact spaces is said to be 1-connected at ∞ if for any pair of neighborhoods (U, V) of ∞ in (X, Y) we may find another $(U', V') \subset (U, V)$ such that any loop $S^1 \subset U'$ can be deformed into V' inside U.

A similar argument yields

1.4. THEOREM. *Let W be a smooth G-manifold, G acting freely on ∂W. Suppose $(W, \partial W)$ is k-connected and* 1*-connected at ∞, and suppose $H^i(W) = 0$ for $i > k$, where* $2 \leqslant k < (n-1)/2$. *Then the conditions*

(1) *W is Smith acyclic, and*

(2) $[\hat{\tau}_W] \in p^* KO(B_G)$

are together necessary and sufficient for the existence of a proper smooth G-embedding $W^n \subset \mathbf{R}^n$ for some smooth action on $\mathbf{R}^n$, free on $\mathbf{R}^n - W$.

The proof of 1.4 is similar to that of 1.2 but we may use infinitely generated modules and an infinite number of handle additions. The condition that $(W, \partial W)$ is 1-connected at ∞ is used to show that the result U of adding all the handles is 1-connected at ∞, so that it is diffeomorphic to $\mathbf{R}^n$. □

1.5. COROLLARY. *Let $G \times M^n \to M^n$, $n \geqslant 6$, be a smooth effective G-action such that* $\dim M^H < (n-1)/2$ *for each nontrivial subgroup $H \subseteq G$. Let $(W, \partial_0 W) \subset (M, \partial M)$ be some smooth G-regular neighborhood of the singular set of M such that $\partial_0 W = \partial M \cap \partial W$ is a G-regular neighborhood of the singular set of ∂M. Then conditions* (1)–(3) *of Theorem* 1.2 *are together necessary and sufficient for the existence of a smooth G-embedding $(W, \partial_0 W) \subset (D^n, S^{n-1})$, with G acting freely on $D^n - W$.*

We will improve this result later.

PROOF. Clearly 1.2(1)–1.2(3) are necessary conditions, so we must prove sufficiency. So it remains to show that if 1.2(1)–1.2(3) are satisfied, together with the hypotheses of 1.4, then the hypotheses of 1.2 are satisfied, namely, that there is a k, $2 \leqslant k < (n-1)/2$, such that

(a) $(W, \partial_1 W)$ *is* k-connected,

(b) $H^i(W) = 0$ for $i > k$, and

(c) $(\partial_0 W, \partial\partial_0 W)$ is 1-connected.

Since $\dim M^H < (n-1)/2$ for all H, $\dim \mathscr{S}(M) < (n-1)/2$, and we set $k =$ greatest integer $< (n-1)/2$, so that $k \geqslant 2$ since $n \geqslant 6$. Then $\dim \mathscr{S}(M) \leqslant k$, so (b) $H^i(\mathscr{S}(M)) = H^i(W) = 0$ for $i > k$. But $\operatorname{codim} \mathscr{S}(M) \leqslant n - k > k + 1$, so (a) $(W, \partial_1 W)$ is k-connected. Finally, $\partial_0 W$ is a G-regular neighborhood of $\mathscr{S}(\partial M) = \mathscr{S}(M) \cap \partial M$, so that $\dim(\partial M) < (n-1)/2 - 1$, and (c) follows. □

1.6. COROLLARY. *Let $G \times M^n \to M^n$ be a smooth effective G-action such that $\dim M^H < (n-1)/2$ for all subgroups $H \neq 1$ of G. Let W be a smooth G-regular neighborhood of $\mathscr{S}(M)$ and let $W_0 = W - \mathscr{S}(\partial W)$. Then conditions* (1) *and* (2) *of Proposition* 1.2 *are necessary and sufficient for the existence of a proper G-embedding of $W_0 \subseteq \mathbf{R}^n$ with G acting on $\mathbf{R}^n$ so that G acts freely on $\mathbf{R}^n - W_0$.*

The result follows from 1.4 by a proof analogous to that of 1.5. □

1.7. THEOREM. *Suppose $G \times W^n \to W^n$ is a smooth effective G-action on a compact W^n such that for some integer k, $2 \leqslant k < (n-1)/2$, $H^i(W) = 0$ for $i > k$, and $\dim W^H < n - k$ for all subgroups $H \neq 1$ of G, and $(W, \partial W)$ 1-connected. Let $\partial_0 W$ be a smooth G-regular neighborhood of $\mathscr{S}(\partial W)$. Then the conditions*

(1) *W is Smith acyclic,*

(2) $[\hat{\tau}_W] \in p^* KO(B_G)$, *and*

(3) $\sigma(W) = 0$

are necessary and sufficient for the existence of a smooth G-embedding $(W, \partial_0 W) \subset (D^n, S^{n-1})$ for some smooth G-action on D^n, free on $D^n - W$.

PROOF. Let $\partial_1 W = \partial W - (\operatorname{int} \partial_0 W)$ as usual, so G acts freely on $\partial_1 W$. Since codimension $\mathscr{S}(\partial W) > 2$, $(\partial W, \partial_1 W)$ is 2-connected so that $(W, \partial_1 W)$ is 1-connected. By 1.3 we find a smooth G-embedding $(W, \partial_0 W) \subset (W_2, \partial W_2)$ such that $W_2 - W$ is G-free and W_2 satisfies all the conditions of 1.7 but W_2 and $\partial_1 W_2$ are 1-connected.

Since $\dim M^n < n - k$ for all $H \neq 1$, $\dim \mathscr{S}(M) < n - k$ and $\dim \mathscr{S}(\partial M) < n - k - 1$. Hence $H^i(\mathscr{S}(\partial M)) = H^i(\partial_0 W) = 0$ for $i \geqslant n - k - 1$, and from the exact sequence of the pair $(W_2, \partial_0 W)$ we deduce that $H^i(W_2, \partial_0 W) \to H^i(W_2)$ is an injection for $i \geqslant n - k$. Since $H^i(W_2) = H^i(W) = 0$ for $i > k$ and $n - k > k$, it follows that $H^i(W_2, \partial_0 W) = 0$ for $i \geqslant n - k$. Since $\partial W_2 = \partial_0 W \cup \partial_1 W_2$ and $\partial_0 W \cap \partial_1 W_2 = \partial\partial_0 W = \partial\partial_1 W_2$, Poincaré-Lefschetz duality implies

$$H_j(W_2, \partial_1 W_2) = 0 \quad \text{for } j \leqslant k,$$

and, since W_2 and $\partial_1 W_2$ are 1-connected, it follows that $(W_2, \partial_1 W_2)$ is k-connected. It is clear that $(\partial_0 W, \partial\partial_0 W)$ is 1-connected since $\partial_0 W$ is a regular neighborhood of a complex of codimension > 2, so the hypotheses of 1.2 are satisfied, which proves 1.7. □

1.8. THEOREM. *Let $G \times W \to W$ be a smooth effective G-action with W and ∂W 1-connected and $(W, \partial W)$ 1-connected at ∞, and such that, for some integer k, $2 \leqslant k < (n-1)/2$, $H^i(W) = 0$ for $i > k$ and* $\dim W^H < n - k$ *for all subgroups $H \neq 1$ in G. Let $W_0 = W - \mathscr{S}(\partial W)$. Then conditions* (1) *and* (2) *of* 1.4,

(1) *W is Smith acyclic, and*

(2) $[\hat{\tau}_W] \in p^*KO(BG)$,

are together necessary and sufficient for the existence of a proper smooth G-embedding $W_0 \subset \mathbf{R}^n$ for some smooth G-action on $\mathbf{R}^n$, free on $\mathbf{R}^n - W_0$. □

1.9. LEMMA. *Let $G \times M \to M$ be a smooth effective Smith acyclic G-action such that* $\dim M^H \leqslant l$ *for all subgroups $H \neq 1$, where M is compact. Then $\mathscr{S}(M) = \mathscr{S}_0(M) \cup (\bigcup_C M^C)$, where C ranges over certain cyclic subgroups of prime order, $M^C - \mathscr{S}_0(M)$ are disjoint,* $\dim \mathscr{S}_0(M) < \dim \mathscr{S}(M)$, *and $\mathscr{S}_0(M)$ is invariant under G.*

PROOF OF 1.9. If $H' \supset H$, then $M^{H'} \subset M^H$. If M^H is connected, then either $\dim M^{H'} < \dim M^H$ or $M^{H'} = M^H$. Any subgroup contains a cyclic subgroup C of prime order and M^C is connected, since the action is Smith acyclic. Thus there are cyclic subgroups $C_1, \ldots, C_k$ of prime order, such that $\dim M^{C_i} = l$ for all i, $M^{C_i} \neq M^{C_j}$ for $i \neq j$, and for any subgroup H, either $M^H = M^{C_i}$ for some i, or $\dim M^H < l$.

Let $\mathscr{S}_0(M) = \bigcup M^H$ such that $\dim M^H < l$, so that $\dim \mathscr{S}_0 < l$ and $\mathscr{S}_0(M)$ is G-invariant. Since $M^{H_1} \cap M^{H_2} = M^{(H_1, H_2)}$, where (H_1, H_2) is the subgroup generated by H_1 and H_2, it follows that $M^{C_i} \cap M^{C_k} \subset \mathscr{S}_0(M)$ if $i \neq j$ so that $M^{C_i} - \mathscr{S}_0(M)$ are disjoint. □

1.10. THEOREM. *Let $G \times M^n \to M^n$ be a smooth effective G-action, M compact, $n \geqslant 6$, and let $(W, \partial_0 W) \subset (M, \partial M)$ be a smooth G-regular neighborhood of the singular set $\mathscr{S}(M)$ in M, with $\partial_0 W$ a smooth G-regular neighborhood of $\mathscr{S}(\partial M)$ in ∂M. Suppose* $\dim \mathscr{S}(M) \leqslant n/2$ *and* $\dim \mathscr{S}_0(M) \leqslant n/2 - 2$. *Then the conditions*

(1) *W is Smith acyclic,*

(2) $[\hat{\tau}_W] \in p^*KO(B_G)$, *and*

(3) $o(W) = 0$ *in* $\tilde{K}_0(\mathbf{Z}G)$

are together necessary and sufficient for the existence of a smooth G-embedding $(W, \partial_0 W) \subset (D^n, S^{n-1})$, with G acting freely on $D^n - W$.

We have that $\dim \mathscr{S}(M) = q \leqslant n/2$, so that $n - (q-1) \leqslant q + 1$, and if we set $k = q - 1$, the hypothesis of 1.7 would be satisfied if we could show that $H^i(\mathscr{S}(M)) = 0$ for $i > q - 1$. This follows from 1.11 below.

1.11. PROPOSITION. *Let $G \times M \to M$ be a smooth effective Smith acyclic G-action such that* $\dim \mathscr{S}(M) \leqslant l$ *and* $\dim \mathscr{S}_0(M) < l - 1$ *($\mathscr{S}_0(M)$ as in* 1.9*). Then the singular set $\mathscr{S}(M)$ collapses equivariantly to a complex L of dimension $\leqslant l - 1$, so that L is a G-deformation retract of $\mathscr{S}(M)$ and $H^i(\mathscr{S}(M)) = H^i(\mathscr{S}(M)/G) = 0$ for $i \geqslant l$.*

To conclude the proof of 1.11 we show that, for each i, M^{C_i} collapses to $L_i = (\mathscr{S}_0 \cap M^{C_i}) \cup$ (disks of dim $< l$). Since the $M^{C_i} - \mathscr{S}_0$ are disjoint, $\bigcup M^{C_i} = \mathscr{S}(M)$ collapses to $\bigcup L_i$ which has dimension $< l$. We must take care to ensure that the collapse is invariant under G to complete the argument.

Let N be the normalizer of C in G, $C = C_i$ cyclic of order p, so that N acts on M^C, freely off $\mathscr{S}_0 \cap M^C$. Take an N-regular neighborhood U^C of $\mathscr{S}_0 \cap M^C$ in M^C, and let $V^C = M^C - (\operatorname{int} U^C)$, so that $\partial V^C = \partial U^C \cup (\partial M^C - U^C)$, and N/C acts freely on V^C. Since ∂M^C is a (mod p)-homology $(l-1)$-sphere by Smith acyclicity, $\partial M^C - U_C$ is a nonempty $(l-1)$-dimensional free N submanifold of ∂M^C. Since $\dim \mathscr{S}_0(M) < l - 1$, $M^C - \mathscr{S}_0(M)$ is connected (for each cyclic subgroup C of prime order).

Hence V^C/N is an l-manifold with $\partial(V^C/N) = (\partial U^C/N) \cup (\partial M^C - U^C)/N$. Choosing an appropriate Morse function on V^C/N gives a representation of V^C/N as $\partial U^C/N \cup$ (handles of index $< l$), since there is another nonempty boundary component. Hence V^C/N collapses to $\partial U^C/N \cup$ (disks of dim $< l$), so V^C collapses N-equivariantly on $\partial U^C \cup$ (disks of dim $< l$). Using the N-equivariant collapse of U^C to $\mathscr{S}_0 \cap M^C$ gives an N-equivariant collapse of M^C to

$$L_i = \mathscr{S}_0 \cap M^C \cup (\text{disks of dim } < l),$$

so $\dim L_i < l$. Using the image of this collapsing on the other disjoint images of M^C gives a G-invariant collapse of GM^{C_i} to $GL_i \subset \mathscr{S}(M)$. Doing this separately for each C_i in different conjugacy classes defines a collapse on all of $\mathscr{S}(M)$ to $L = \bigcup L_i$. □

1.12. REMARK. If each $g \in G$ preserves orientation of M^C, where $|C| =$ prime, then $\dim \mathscr{S}_0(M) \leqslant \dim \mathscr{S}(M) - 2$ and 1.11 holds for such actions. The conclusion of 1.11 is also true if G is a p-group, or if the action is semifree.

2. Cohomological conditions. In this section we give some sufficient conditions in terms of group cohomology which imply the tangential condition of §1, and another which implies Smith acyclicity. We give a simple geometric application to semifree actions, generalizing [**J**].

2.1. THEOREM. *Let $G \times W^n \to W^n$ be a smooth effective G-action, W a stably parallelizable manifold. If $H^p(G; \widetilde{KO}^*(W)) = 0$ for $p > 0$, then $[\hat{\tau}_W] \in p^*\widetilde{KO}^*(B_G)$, where $\widetilde{KO}^*(W)$ is made a $\mathbf{Z}G$-module via the action.*

PROOF. If we consider the spectral sequence of Atiyah, Hirzebruch, Serre and Moore for the pair of fibre spaces $(B_G, \hat{W})$ over B_G (considering B_G as a fibre space over B_G with contractible fibre C in which W is embedded),

$$E_2^{p,q} = H^p(B_G;\, KO^q(C, W)),$$

converging to the associated graded group of $KO^*(B_G, \hat{W})$. Since C is contractible, $KO^q(C, W) \cong \widetilde{KO}^{q-1}(W)$; so by hypothesis

$$E_2^{p,q} = \begin{cases} 0 & \text{for } p > 0, \\ \widetilde{KO}^{q-1}(W)^G & \text{for } p = 0. \end{cases}$$

Thus $E_2 = E_\infty$, $KO^q(B_G, W) = \widetilde{KO}^{q-1}(W)^G$, and the exact sequence of the map p: $W \to B_G$ becomes:

$$\begin{array}{ccccccccc} \cdots & \to & \widetilde{KO}^{q-1}(W)^G & \to & \widetilde{KO}^q(B_G) & \to & {}^*\widetilde{KO}^q(\hat{W}) & \xrightarrow{j^*} & \widetilde{KO}^q(W)^G & \to & \cdots \\ & & & & & & & \searrow & \cap & & \\ & & & & & & & & \widetilde{KO}^*(W) & & \end{array}$$

Since W is stably parallelizable, $j^*[\tau_W] = 0$ in $\widetilde{KO}^*(W)$; so $[\hat{\tau}_W] \in p^*KO(B_G)$. □

2.2. COROLLARY. *Let W^n be a stably parallelizable compact smooth submanifold $n \geqslant 6$ and let $G \times W \to W$ be an effective action such that*

(1) *W is Smith acyclic,*

(2) $\dim W^H < n/2$ *for all subgroups $H \neq 1$ in G, and*

(3) $\bar{H}^*(G; \widetilde{KO}^*(W)) = 0$.

Then there is a smooth action of G on D^n whose restriction to a neighborhood of the singular set agrees with a neighborhood of the singular set of W if and only if $\sigma(W) = 0$ in $\widetilde{KO}_0(\mathbf{Z}G)$.

This follows immediately from 2.1, together with 1.8.

Other results of §1 may be similarly adapted to this hypothesis. See 1.8, 1.10 and 2.12.

We may apply 2.2 to semifree actions of G on disks, generalizing [**J**, Theorem 2.1]. Note that this theorem characterizes the stationary point set of semifree actions on the disk in this dimension range.

2.3. THEOREM. *Let $(F^k, \partial F^k) \subset (D^n, \partial D^n)$ be a submanifold with normal bundle ν, where $k \leqslant n/2$ and $n \geqslant 6$. Then the following conditions together are necessary and sufficient for existence of a semifree G-action on D^n with stationary point set F:*

(i) $\bar{H}_*(F\colon \mathbf{Z}/q\mathbf{Z}) = 0$, *where $q = |G|$.*

(ii) *ν admits the structure of G-bundle leaving the zero section fixed and having a free representation on each fibre.*

(iii) $\sigma(F) = \Sigma_{i>0}(-1)^i\sigma_G(H_i(F)) \in \tilde{K}_0(\mathbf{Z}G)$ *vanishes.*

PROOF. This follows from Corollary 2.1, 1.10 and 1.12. Condition (i) implies that $\tilde{KO}^*(F)$ is torsion prime to $|G|$, using the Atiyah-Hirzebruch spectral sequence; hence $\bar{H}^*(G; \tilde{KO}^*(F)) = 0$. Clearly the total space $E(\nu)$ is Smith acyclic. □

2.4. REMARK. The condition $\bar{H}^*(G; \widetilde{KO}^*(W)) = 0$ does not imply that $\widetilde{KO}^*(W)$ is torsion prime to $|G|$. Suppose for example that W is Smith acyclic and $(k-1)$-connected, with $H^i(W) = 0$ for $i > k$, so that $H_k(W)$ is $\mathbf{Z}$-free. Then $H_k(W)$ is $\mathbf{Z}G$-projective. Since $\bar{H}^i(W) = 0$ for $i \neq k$, the Atiyah-Hirzebruch spectral sequence degenerates and $\widetilde{KO}^*(W) \cong \bar{H}^k(W) \otimes \widetilde{KO}^*(\text{pt.})$ as a $\mathbf{Z}G$-module and hence is cohomologically trivial, and, in particular, $\bar{H}^*(G; \widetilde{KO}^*(W)) = 0$.

It is interesting to note the condition of Smith acyclic is implied by a cohomological condition. Let $\hat{H}^*(G; M)$ be the Tate cohomology (complete cohomology in the usage of [Cartan and Eilenberg]) of the group G with coefficients in the $\mathbf{Z}G$-module M. A $\mathbf{Z}G$-module is said to be *cohomologically trivial* if $\hat{H}^*(K; M) = 0$ for every subgroup $K \neq 1$ of G (including $K = G$).

2.5. THEOREM. *Let X be a finite-dimensional G-CW-complex such that $\overline{H}_*(X)$ is a cohomologically trivial* $\mathbf{Z}G$*-module. Then X is Smith acyclic.*

2.6. LEMMA. *Let (X, X_0) be a K-pair such that $H_i(X, X_0) = 0$ for $i \neq k$, $H_k(X, X_0)$* $\mathbf{Z}K$*-projective, and $H_i(X) = 0$ for $i < k$.*
If $\hat{H}^*(K; \overline{H}_*(X_0)) = 0$, then $\hat{H}^*(K; \overline{H}_*(X)) = 0$.

PROOF. Let $F = H_k(X, X_0)$ and let $A = \text{image}\, H_k(X) \subset H_k(X, X_0)$. Then we have exact sequences

$$(\alpha) \quad 0 \to A \to F \to H_{k-1}(X_0) \to 0,$$
$$(\beta) \quad 0 \to H_k(X_0) \to H_k(X) \to A \to 0.$$

From (α) we get a long exact sequence

$$\cdots \to \hat{H}^*(G; A) \to \hat{H}^*(G; F) \to \hat{H}^*(G; H_{k-1}(X_0)) \to \cdots.$$

But $\hat{H}^*(G; F) = 0$ since F is $\mathbf{Z}G$-projective, so $\hat{H}^*(G; H_{k-1}(X_0)) \cong H^{*-1}(G; A) = 0$. From (β) we get a long exact sequence

$$\cdots \to \hat{H}^*(G; H_k(X_0)) \to \hat{H}^*(G; H_k(X)) \to \hat{H}^*(G; A) \to \cdots,$$

which implies $\hat{H}^*(G; H_k(X)) = 0$. For $i > k$, $H_i(X) \cong H_i(X_0)$, so the lemma follows. □

2.7. LEMMA. *Suppose X is a finite-dimensional G-space with $\overline{H}^*(X)$ cohomologically trivial over G. Then there is a G-embedding $X \subset X_k$ for some k such that*
(1) *$X_k - X$ is G-free*, $\dim(X_k - X) \leqslant k$,
(2) *X_k is $(k-1)$-connected, and*
(3) *$\overline{H}_*(X_k)$ is a cohomologically trivial* $\mathbf{Z}G$*-module.*

PROOF. Suppose true for k by induction. Let $\{h_\alpha\} \in \pi_k(X_k)$ be a set of generators for $\pi_k(X_k)$ over $\mathbf{Z}G$ and let $X_{k+1} = X_k \cup_{gh\alpha} \bigcup_\alpha D_\alpha^{k+1}$, adding free G-cells on each h. Then $X_{k+1} - X_k$ is G-free of dimension $k+1$ and X_{k+1} is k-connected, and it remains to show (3). Since $X_{k+1} - X_k = \bigcup GD^{k+1}$, a disjoint union of free G-cells, $H_i(X_{k+1}, X_k) = 0$ for $i \neq k+1$ and $H_{k+1}(X_{k+1}, X_k)$ is $\mathbf{Z}H$-free for any $H \subset G$, so, by 2.6 applied to each subgroup $K \subseteq G$, $H_*(X_k)$ is a cohomologically trivial $\mathbf{Z}G$-module. □

Taking $k > \dim X$ and applying 2.7, we get $X \subset X_k$, with $\dim X_k = k$, $X_k (k-1)$-connected, $X_k - X$ G-free, and $\hat{H}^*(H; \overline{H}_*(X_k)) = \hat{H}^*(H; \overline{H}_k(X)) = 0$. Since X_k is $(k-1)$-connected and k-dimensional, $H_k(X_k)$ is $\mathbf{Z}$-free; so, by [**R**, (4.11)], $H_k(X_k)$ is $\mathbf{Z}G$-projective.

There is a $\mathbf{Z}G$-free module M (infinitely generated) such that $H_k(X_k) + M$ is free over $\mathbf{Z}G$. Attaching an infinite number of G-free k-disks to X_k we get $X_k' \supset X_k$ with the same properties but with $H_k(X_k')$ $\mathbf{Z}G$-free. We may then attach G-free $(k+1)$-disks to X_k' to get Y with $\overline{H}_*(Y) = 0$, $Y - X$ G-free, so that $\mathscr{S}(Y) = \mathscr{S}(X)$. Hence X is Smith acyclic, since Y is acyclic.

2.8. REMARK. Note that cohomologically trivial is stronger than Smith acyclic.

3. Classifying actions on disks. Let us consider G-actions on D^n which extend a given action on a regular neighborhood of the singular set. We will show that if the singular set has dimension $< [n/2]$ ($=$ greatest $m \in \mathbf{Z}$, $m \leqslant n/2$), such actions are classified by elements of the Whitehead group $\mathrm{Wh}(G)$.

More precisely, let $G \times N^n \to N^n$ be a smooth G-action, N a compact regular neighborhood of $\mathscr{S}(N)$. Suppose ϕ_i: $G \times D_i^n \to D_i^n$, $i = 1, 2$, are smooth G-actions with N embedded in each, such that G acts freely on $D_i^n - N$.

3.1. THEOREM. *Suppose* $\dim \mathscr{S}(N) < [n/2]$. *Then the set of G-extensions of N to* D^n, *modulo diffeomorphisms which are the identity on N, is in one-one correspondence with the Whitehead group* $\mathrm{Wh}(G)$.

One may interpret 3.1 slightly differently in this equivalent formulation:

(3.2) Suppose $\dim \mathscr{S}(D_i^n) < [n/2]$, and let N_i be smooth G-regular neighborhoods of $\mathscr{S}(D_i^n)$, $i = 1, 2$. A G-diffeomorphism f: $N_1 \to N_2$ defines an element $W(f) \in \mathrm{Wh}(G)$ such that f extends to F: $D_1^n \to D_2^n$ if and only if $W(f) = 0$.

Given $G \times N \to N$ as in 3.1 there is an action $G \times D_0^n \to D_0^n$ from 1.2 such that $D_0^n = N \cup$ (handles of index $\leqslant k$), where $n = 2k + 3$ or $n = 2k + 2$ (compare (1.2) and (1.10)).

3.3. PROPOSITION. *With the hypothesis of* (3.2), *there are smooth G-embeddings* f_i: $D_0^n \to D_i^n$, $i = 1, 2$ (*and* D_0 *as above*) *with* $f_i(N) = N_i$, $f_i(D_0^n)$ *are unique up to isotopy, and such that* $ff_1|N = f_2|N$.

PROOF. Let $A_i =$ the closure of $(D_i^n - N_i)/G$, $\partial_+ A_i = \partial A_i \cap N_i/G$, and let e_i: $\partial_+ A_0 \to A_i$ be induced by the restriction of f and the inclusion $\partial_+ A_i \subset A_i$. Consider the following commutative diagram in which c_0, c_i are the classifying maps of the associated covering spaces:

$$\begin{array}{ccccc} & & A_i & & \\ & e_i \nearrow & & \searrow c_i & \\ \partial_+ A_0 & & & & BG \\ & \text{incl} \searrow & & \nearrow c_0 & \\ & & A_0 & & \end{array}$$

We claim that there is a map f_i': $A_0 \to A_i$ which extends e_i and makes the above diagram (homotopy) commutative. To see this, observe that the obstructions for existence of f' lie in

$$H^{q+1}\big(A_0, \partial_+ A_0; \{\pi_q(F_i)\}\big) \quad (\text{for } i \geqslant 2),$$

where F_i is the homotopy fibre of c_i: $A_i \to BG$ and $\{\pi_q(F_i)\}$ is the associated local system. For $q \geqslant 2$, $\pi_q(F_i) \cong \pi_q(A_i)$ as a G-module. Now A_0 is obtained from $\partial_+ A_0$ by attaching cells of dimension less than $n/2$ so $H^{q+1}(A_0, \partial_+ A_0; \{\pi_q(A_i)\}) = 0$ for $q > n/2 - 1$. Let q be the smallest integer > 1 such that $\pi_q(A_i) \neq 0$. Then $\pi_q(A_i) \cong H_q(D_i^n - N_i) \cong H^{n-q-1}(N_i)$ by Alexander duality, and the latter group is zero for $q \leqslant n/2 - 1$, since $\dim \mathscr{S}(N_i) < n/2$ by assumption. Thus, all the obstructions vanish and f_i' exists. It follows from the comparison of the Mayer-Vietoris sequences of $N_i \cap \tilde{A}_i$ and the five-lemma that f_i' is a homotopy equivalence. Furthermore, f_i' is tangential by the commutativity of the above diagram and

3.4. PROPOSITION. *Let (D^n, ϕ_i) $(i = 1, 2)$ be two G-disks such that the G-regular neighborhoods of their singular sets, say N_i $(i = 1, 2)$, are G-diffeomorphic. Let $A_i = (D^n - \text{int}(N_i))/G$ and c_i: $A_i \to BG$ be the classifying maps of the associated covering spaces. Then, there is a (stable) bundle ξ over BG such that $\tau A_i = c_i^*\xi$ (stably).*

PROOF. We proceed in several steps:

If $x \in D^n$ is stationary, it has a tangential linear representation ρ and also defines a section S: $B_G \to \hat{D}^n$ $(\hat{D}^n = D^n \times_G E_G)$. Clearly, $S^*(\tau_{D^n}) = \bar{\rho}$ where $\bar{\rho}$ is the n-linear bundle over B_G induced by the representation ρ. Since S is a homotopy equivalence and $x \in N$, the result follows in this case.

If $H \subset G$ is a p-group, then H has a stationary point on D^n, using Smith theory. Moreover, the tangential representation is unique because $(D^n)^H$ is connected. Hence, the restrictions to every p-subgroup of G for $\hat{\tau}_{D_i}$ are the same, so the elements in $KO(B_G)$ are the same (see [**At**, pp. 46–48]) and this completes the proof of 3.4. □

To complete the proof of 3.3 we first find a G-immersion f_i^0: $D_0^n \to D_i^n$ homotopic to f_i using f on N and the Smale-Hirsch immersion theorem using 3.4. Since $D_0 = N \cup$ (handles of index $\leqslant k$), $k < [n/2]$, it follows from a general position statement that the embedding of N may be extended to an embedding on each handle, unique to isotopy. See Wall [**W2**]. □

The region $D_i^n - f_i(\mathring{D}_0^n)$ is an h-cobordism of free G-manifolds from $f_i(\partial_+ A_0)$ to $\partial_- A_i$ $(\partial A_i = \partial_+ A_i \cup \partial_- A_i)$. Let x_i = Whitehead torsion of this h-cobordism. Then the diffeomorphism $f_2 f_1^{-1}$: $f_1(\partial_+ A_0) \to f_2(\partial_+ A_0)$ extends to a diffeomorphism if and only if $x_1 = x_2$ (see [**M2**]), which completes the proofs of 3.2 and 3.1. □

The classification of smooth G-actions on S^n can be reduced to the classification of G-disks, and understanding of the structure of Rothenberg's class of "semilinear" G-spheres, provided that $(S^n)^G \neq 0$. (See [**RS**] for definitions and properties of semilinear disks and spheres.)

3.6. THEOREM. *Let (S^n, ϕ_1), $i = 1, 2$, be two G-spheres such that* $\dim(S^n, \phi_1)^H <$ $[n/2]$ *for all $1 \neq H \subseteq G$, and $(S^n, \phi_i)^G \neq 0$. Suppose that equivariant regular neighborhoods of the singular sets of (S^n, ϕ_i) are G-diffeomorphic. Then there is a semilinear G-sphere (S^n, ψ) such that (S^n, ϕ_2) is G-diffeomorphic to the equivariant connected sum $(S^n, \phi_1) \# (S^n, \psi)$.*

PROOF. Let $x \in (S^n, \phi_1)^G$ and choose a linear disk neighborhood $D^n(x) \subset (S^n, \phi_1)$. Then the proof of Proposition 3.3 yields an equivariant embedding f: $(D_0^n, \phi_1) \to (S^n, \phi_2)$, where $(D_0^n, \phi_1) = (S^n, \phi_1) - \text{int}(D^n(x), \phi | D^n(x))$. The desired semilinear sphere is

$$(S^n, \psi) = \big((S^n, \phi_2) - \text{int}\, f(D_0^n, \phi_1) \cup (D^n(x), \phi_1 D^n(x)\big).$$

Clearly $(S^n, \phi_2) = (S^n, \phi_1) \# (S^n, \psi)$. □

The G-diffeomorphism classes of the semilinear disks (S^n, ψ), with $(S^n, \psi)^G \neq 0$ and having a prescribed linear representation ρ as their model (i.e., the G-tangent space at each $x \in (S^n, \psi)^G$ is equivalent to ρ), form a group under equivariant connected sum, provided that we have the above-mentioned stability condition and a further condition that the codimension of adjacent fixed point sets be larger than two. Then, it is possible to define an action of the "group of semilinear spheres" $R(\rho)$ on the set of G-diffeomorphism classes of G-spheres with ρ appearing as the tangential representation at some $x \in (S^n)^G$, and each orbit of the action is characterized by the G-diffeomorphism class of its singular set.

Call two actions $G \times D_i^n \to D_i^n$, $i = 0, 1$, 'concordant' if there is an action

$$G \times (D^n \times [0,1]) \to D^n \times [0,1]$$

such that the action restricted to $D^n \times i$ is equivalent to D_i^n. Clearly, for a concordance, a G-regular neighborhood U^{n+1} of $\mathscr{S}(D^n \times [0,1])$ is a Smith acyclic G-manifold with $[\hat{\tau}_u] \in p^*KO(B_G)$ and $\sigma(U) = 0$, since $D^n \times [0,1]$ is diffeomorphic (except at corners) to D^{n+1}. Conversely, if U is a G-manifold (with corners) such that $\partial U = N_0 \cup V \cup N_1$, where N_i are G-regular neighborhoods of (D_i^n), $i = 0, 1$, then the results of §1 (for example 1.2), show that under appropriate dimension restrictions, etc., these three conditions (Smith acyclicity, the tangential condition and the projective class condition) are necessary and sufficient for extending U to a concordance of D_0^n to D_1^n, by extending the given action on $D_0^n \times [0, \varepsilon] \cup U \cup D_1^n \times [1-\varepsilon, 1]$ to an action on the disk $D^n \times [0,1]$.

4. A counterexample. In this section, we discuss Jones' Theorem [**J**] and Theorem 2.1, and we will see by means of an example that the tangential data referred to in his proof are not sufficient. We are informed that the original proof in Jones' thesis was correct.

4.1. EXAMPLE. Let $\mathbf{Z}/2\mathbf{Z}$ act on S^n linearly and let $p\colon S^n \to \mathbf{R}P^n$ in which the k-skeleton is $\mathbf{R}P^k$. This yields the $\mathbf{Z}/2\mathbf{Z}$-equivariant cellular subdivision of S^n. Consider $S^1 \subset S^n$ and $p(S^1) \cong S^1 \subset \mathbf{R}P^n$, so that $p\colon S^1 \to S^1$ is a map of degree 2. We take n to be an odd number, so that $\mathbf{R}P^n$ is orientable and $\tau(\mathbf{R}P^n)|S^1$ admits a framing ψ, where $\tau(X) =$ stable tangent bundle of X. Since $\pi_1 SO(m) \cong \mathbf{Z}/2\mathbf{Z}$, $m > 3$, we notice that the induced framing $p^*\psi$ on $\tau(S^n)|S^1$ is independent of choice of ψ. Now let ϕ be any framing of the stable tangent bundle of S^n. The construction of Jones [**J**, pp. 62, 63] calls for an "appropriate choice" of framing ψ of $\tau(\mathbf{R}P^n)|S^1$ so that $\phi S^1 \simeq p^*\psi$, i.e., it is claimed that ϕ can be deformed to a new framing ϕ' such that $\phi'|S^1$ is equivariant with respect to the $\mathbf{Z}/2\mathbf{Z}$-action. This is the simplest possible situation in which S^n, with the linear $\mathbf{Z}/2\mathbf{Z}$-action, may be regarded as the normal sphere bundle to the fixed point set $(F^k, \partial F^k) \subset_\nu (D^{n+1+k}, S^{n+k})$, as is the case in the first step of the inductive proof of [**J**].

We claim, however, that it is impossible to find such $\phi' \simeq \phi$ with $\phi'|S^1$ an equivariant framing, i.e., for any choice of ψ and ϕ, $\phi|S^1 \simeq p^*\psi$ is impossible, for the case $n \equiv 1 \pmod 4$.

Suppose the contrary and let ϕ' be a framing such that $\phi'|S^1$ is equivariant. Then we show that ϕ' can be made homotopic to a framing ϕ'' of $\tau(S^n)$ such that $\phi''|S^2$ is equivariant (S^2 is the equivariant 2-skeleton of S^n). Let $g\colon S^n \to S^n$ be the given involution and let $S^2 = D^2_+ \cup D^2_-$, where $D^2_+ \cap D^2_- = S_1$ and $g(D^2_\pm) = D^2_\mp$. The induced framing $g^*\phi'$ and ϕ' agree on S^1. We fix $\phi'|D^2_+$ and compare $g^*\phi'|D^2_-$. This yields a map $\Delta = \Delta(\phi'|D^2_-, g^*\phi'|D^2_-): D^2_- \to SO(m)$, such that $\Delta(S^1) = 1 \in SO(m)$, the identity element, since $\phi'|S^1$. Since $\pi_2 SO(m) = 0$, Δ is null-homotopic. Hence we can deform $\phi'|D^2_-$ in a small neighborhood of D^2_- rel S^1 (and apply the homotopy extension property) to obtain a new framing ϕ'' of $\tau(S^n)$ such that $\phi''|D^2_+ = \phi'|D^2_+$ and $\phi''|D^2_- = g^*\phi'|D^2_-$, i.e., $\phi''|S^2$ is equivariant with respect to the $\mathbf{Z}/2\mathbf{Z}$-action, and $\phi'' \simeq \phi'$ by our choice. Hence we have an induced framing ϕ'_* of $\tau(\mathbf{R}P^n)|\mathbf{R}P^2$, where $\mathbf{R}P^2 = p(S^2)$ is the 2-skeleton of $\mathbf{R}P^n$. But the second Stiefel-Whitney class $w_2(\tau(\mathbf{R}P^n)|\mathbf{R}P^2) \neq 0$ when $n \equiv 1 \pmod 4$. This provides the desired counterexample.

Clearly, the modification of this argument applies to any periodic homeomorphism (semifree or not) on a regular neighborhood $N(F^k) \subset D^{n+k+1}$ with $n \equiv 1 \pmod 4$. Moreover, it becomes evident that the hypothesis $w_2(\rho) = 0$ is necessary for such framings to exist. The claim of Jones, in the last paragraph of [**J**, p. 62], will work precisely when we assume that the second Stiefel-Whitney class of the orbit space (i.e., the complement of the fixed point set) vanishes. Moreover, his claim in the first paragraph of [**J**, p. 63], regarding the change of framing, also breaks down for similar reasons.

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